Fourier theory and applications

Mathematical Analysis part of Mathematical methods

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Preface

These are the lecture notes for the Mathematical Analysis part of the course Mathematical Methods. I will try to timely update the notes as the course proceeds.

Prerequisites for the course are a good mastering of Linear Algebra and of Differential and Integral Calculus. Most of you are probably familiar with the basics of much of the material which is covered here, but we will go anyway through theory and applications from scratch.

Fourier theory and transforms are more than a tool in Engineering. They are the language in which much of our understanding of the physical world is written, and, as a consequence, of the systems we devise to exploit its laws. The basic notion underlying the theory is that a signal can be expressed both as a function of time and of frequency, and that the information it carries can be read off some combination of both. During the course we will go through the mathematical aspects of the theory, but we will mention just few of the interpretations and applications, which will be the subject of more specialized courses.

In the time of high performance computers, automated symbolic calculus, and AI, the computational skills of engineers are less crucial than in the past. What is more relevant is a good understanding of the theory and of ways in which this can be used to model reality and designining devices. The theory, however, is just a dead body of statements if one is not able to recognize its presence in a concrete occurrence of it, its relevance to the solution of a rel-life problem, its usefulness in splitting a complex problem in subproblems which are more amenable to solution. For these reasons, examples and some simple exercises are crucial to the useful understanding of the material.

Contents

CHAPTER 1

Background material

1. Complex numbers

1.1. Basic properties. A *complex number* is one having the form $z = x + iy$, where $x, y \in \mathbb{R}$ are real and the symbol i, the *imaginary unit*, satisfies:

$$i^2 = -1. \tag{1.1}$$

The set of the complex numbers is denoted by $\mathbb{C}$. The map $L : z = x + iy \mapsto (x, y)$ is a bijection $\mathbb{C} \xrightarrow{L} \mathbb{R}^2$. We usefully view $\mathbb{R}^2$ as a plane endowed with Cartesian coordinates (x, y). The same way, we can look at $\mathbb{C}$ as the *complex plane*, endowed with a single *complex* coordinate z. We have $\mathbb{R} \subset \mathbb{C}$: $\mathbb{R}$ is the *real axis* in the complex plane. The *imaginary axis* is

$$i\mathbb{R} = \{iy : y \in \mathbb{R}\}.$$

If $z = x + iy$ and $w = u + iv$, their sum and product are carried out in the natural way:

$$z + w = (x + iy) + (u + iv) = (x + u) + i(y + v), \tag{1.2}$$

$$zw = (x + iy) \cdot (u + iv) = xu + xiv + iyu + i^2yv = (xu - yv) + i(xv + yu). \tag{1.3}$$

If $z = x + iy \in \mathbb{C}$, we say that $x = \mathrm{Re}z$ is the *real part* of z and that $y = \mathrm{Im}z$ is the *imaginary part* of z. Also,

$$\overline{z} = x - iy \tag{1.4}$$

is the *complex conjugate* of z and

$$|z| = (x^2 + y^2)^{1/2} \tag{1.5}$$

is the *modulus* of z. Observe that

$$z\overline{z} = |z|^2. \tag{1.6}$$

Sum and product satisfy the usual algebraic properties.

(1) $+$ is *associative*

$$(z + w) + \zeta = z + (w + \zeta)$$

for all z, w, ζ in $\mathbb{C}$;

(2) $\cdot$ is *associative*

$$(zw)\zeta = z(w\zeta)$$

for all z, w, ζ in $\mathbb{C}$;

(3) $+$ is *commutative*

$$z + w = w + z$$

for all z, w in $\mathbb{C}$;

(4) $\cdot$ is *commutative*

$$zw = wz$$

for all z, w in $\mathbb{C}$;

(5) $0 = 0 + i0$ is the *identity element of the sum*,

$$z + 0 = 0 + z = z$$

for all z in $\mathbb{C}$;

(6) $1 = 1 + i0$ is the *identity of the product*,

$$z1 = 1z = z$$

for all z in $\mathbb{C}$;

(7) $-z = -(x + iy) = (-x) + i(-y)$ is the *opposite* (the *inverse element* with respect to the sum) of z,

$$z + (-z) = (-z) + z = 0;$$

(8) if $z \neq 0$, then $z^{-1} := \frac{\bar{z}}{|z|^2}$ is the *reciprocal* (the *inverse element* with respect to the product) of z,

$$z \cdot z^{-1} = z^{-1} \cdot z = 1;$$

(9) the product is *distributive* with respect to the sum,

$$(z + w)\zeta = z\zeta + w\zeta$$

for all z, w, ζ in $\mathbb{C}$.

The existence of the reciprocal guarantees the existence of ratios,

$$\frac{z}{w} := zw^{-1},$$

provided $w \neq 0$.

These properties are what is need to carry out all of the usual algebra, including introducing polynomials and their operations (sum, product, division with remainder, GCD and LCM...), and rational functions. We list below some properties which are more specific to $\mathbb{C}$.

(1) *Properties of the conjugate*:

$$\overline{z + w} = \bar{z} + \bar{w},\ \overline{zw} = \bar{z}\bar{w},\ \bar{\bar{z}} = z,\ \overline{z : w} = \bar{z} : \bar{w}\ (\text{ if } w \neq 0).$$

Also, $z = \bar{z}$ if and only if $z \in \mathbb{R}$.

(2) *Properties of the modulus*:

$$|zw| = |z| \cdot |w|,\ |z : w| = |z| : |w|\ (\text{ if } w \neq 0).$$

For the sum things are different,

$$|z + w| \leq |z| + |w|,\ |z - w| \geq ||z| - |w||.$$

(3) *Properties of real and imaginary parts*:

$$\text{Re}(z + w) = \text{Re}(z) + \text{Re}(w),\ \text{Im}(z + w) = \text{Im}(z) + \text{Im}(w).$$

1.2. Exponentials in $\mathbb{C}$ and the exponential expression of a complex number. Let $y \in \mathbb{R}$. We define

$$e^{iy} = \cos(y) + i\sin(y). \tag{1.7}$$

This is just a definition, but it is not arbitrary. From the expression $\varphi(y) = e^{iy}$ we expect at least two properties,

$$\varphi(y_1 + y_2) = \varphi(y_1)\varphi(y_2), \text{ and } \varphi'(y) = i\varphi(y). \tag{1.8}$$

Let's verify both. For the first,

$$\begin{aligned} e^{i(y_1+y_2)} &= \cos(y_1 + y_2) + i\sin(y_1 + y_2) \\ &= \cos(y_1)\cos(y_2) - \sin(y_1)\sin(y_2) + i[\cos(y_1)\sin(y_2) - \sin(y_1)\cos(y_2)] \\ &= [\cos(y_1) + i\sin(y_1)] \cdot [\cos(y_2) + i\sin(y_2)] \\ &= e^{iy_1}e^{iy_2}. \end{aligned}$$

For the second,

$$\begin{aligned} \frac{de^{iy}}{dy} &= \frac{d}{dy}[\cos(y) + i\sin(y)] \\ &= \frac{d}{dy}\cos(y) + i\frac{d}{dy}\sin(y) \\ &= -\sin(y) + i\cos(y) \\ &= i[\cos(y) + i\sin(y)] \\ &= ie^{iy}. \end{aligned}$$

More generally, we can define

$$e^z = e^{x+iy} = e^x e^{iy} = e^x[\cos(y) + i\sin(y)].$$

We still have

$$e^{z+w} = e^z e^w, \text{ and } e^1 = e$$

the usual Neper constant.

Any complex number $z = x + iy \neq 0$ can be written in *exponential form*,

$$z = e^{s+it} = e^s[\cos(t) + i\sin(t)], \tag{1.9}$$

with $s, t \in \mathbb{R}$. Let's review how. Equation (1.9) can be written as (assuming for the moment that $x \neq 0$)

$$\begin{aligned} &\frac{y}{x} = \frac{\sin(t)}{\cos(t)} = \tan(t), \text{ with } y \text{ and } \sin(t) \text{ having the same sign,} \\ &|z| = e^s. \end{aligned}$$

We immediately get $s = \ln|z|$. For t we obtain

$$t = \begin{cases} \arctan(y/x) + 2k\pi \text{ if } x > 0; \\ \arctan(y/x) + \pi + 2k\pi \text{ if } x < 0; \\ \pi/2 + 2k\pi \text{ if } x = 0 \text{ and } y > 0; \\ -\pi/2 + 2k\pi \text{ if } x = 0 \text{ and } y < 0, \end{cases}$$

where $k \in \mathbb{Z}$ can be any integer, positive or negative. We say that t is an *argument* of z.

A slight variation on the exponential form is the *trigonometric form* of a complex number z,

$$z = r[\cos(t) + i\sin(t)]. \tag{1.10}$$

Clearly, $r = |z|$ and t is an argument of z, if $z \neq 0$, and can be any real number if $z = 0$.

1.3. The roots of unity. The equation

$$z^N = 1, \tag{1.11}$$

with $N \geq 1$, has N roots $z_j = e^{2\pi ij/N}$, $j = 0, 1, \dots, N-1$. They satisfy the relations

$$z_j z_k = z_{j+k},\ \overline{z_j} = z_{-j} = z_j^{-1},\ z_0 = 1,\ |z_j| = 1. \tag{1.12}$$

Let $G_N = \{e^{2\pi ij/N} : j = 0, 1, \dots, N-1\} \subset \mathbb{C}$. Then, G_N has N elements and it is closed under multiplication.

Since $z^N - 1 = (z-1)(1 + z + z^2 + \cdots + z^{N-1})$, we have that

$$0 = \sum_{k=0}^{N-1} z_j^k = \sum_{k=0}^{N-1} e^{2\pi ijk/N} \tag{1.13}$$

for $j = 1, \dots, N-1$.

The relations (1.12) suggest to introduce the group $\mathbb{Z}_N$ of the *residues modulo* N, which is a periodicization of $\mathbb{Z}$. For j in $\mathbb{Z}$, let

(1.14)

$$[j]_N := \{n \in \mathbb{Z} :\ n - j \text{ is divisible by } N\} = \{n \in \mathbb{Z} : n = qN + j \text{ for some } q \in \mathbb{Z}\}.$$

The set

You can think of an analog clock as a model for $\mathbb{Z}_{12}$. As a set,

$$\mathbb{Z}_N = \{[0]_N, [1]_N, \dots, [N-1]_N\},$$

where we use the notation $[j]_N$ to highlight that $[j]_N$ *is not* one of the usual integers. The successors in $\mathbb{Z}_N$ are defined by:

$$\begin{aligned} [0]_N + [1]_N &= [1]_N \\ [1]_N + [1]_N &= [2]_N \\ &\cdots \\ [N-1]_N + [1]_N &= [0]_N. \end{aligned}$$

You might think of $[0]_{12}$ as playing the role of midnight (or noon) in the analog clock $\mathbb{Z}_{12}$.

The sum can be extended to an operation $\mathbb{Z}_N \times \mathbb{Z}_N \xrightarrow{+} \mathbb{Z}_N$ in the natural way, $[j]_N + [k]_N = [j]_N + [1]_N + \cdots + [1]_N$, where we perform k sums. By default, $[j]_N + [0]_N = [j]_N$. This sum has the usual properties of a sum: it's associative, commutative, it has $[0]_N$ as the zero element, each $[j]_N$ has inverse $[N-j]_N$ ($[j]_N + [N-j]_N = [0]_N$).

Time can be encoded on an analog clock, at the expenses of a loss of information: the clock says it's 4, but it does not specify if it's day or night, and which day. Consider the map $\mathbb{Z} \xrightarrow{\gamma} \mathbb{Z}_N$,

$$\gamma(n) = [j]_N \text{ if and only if } n = qN + j \text{ for some } q \text{ in } \mathbb{Z}, \tag{1.15}$$

i.e. if j is the *remainder* of the division $n : N$. We may interpret $[j]_N$ as a subset of $\mathbb{Z}$,

$$[j]_N = \{n : \ n = qN + j \text{ for some } q \text{ in } \mathbb{Z}\},$$

which is the class of the integers having *remainder* q with *modulus* N.

REMARK 1.1. *For* $n, m \in \mathbb{Z}$,

$$\gamma(n) = \gamma(m) \text{ if and only if } e^{2\pi in/N} = e^{2\pi in/N}.$$

Also,

$$\gamma(n + m) = \gamma(n)\gamma(m).$$

In many applications the point of view is that $\mathbb{Z}_N$ encodes *discrete, finite (periodic) time*. We might also view $[0].[1], \ldots, [N-1]$ as indices of an array. What is specific of $\mathbb{Z}_N$ is that, when we shift the elements of the array by a step forward, the last element of the original array becomes the first of the shifted one, as when turning a gear wheel.

1.4. Exercises.

EXERCISE 1.2. Decompose $z^2 + 1$ as the product of two polynomials with complex coefficients of degree one with respect to the variable z.

EXERCISE 1.3. Show that if x, y, u, v are real numbers, then

$$(xu + yv)^2 \leq (x^2 + y^2)(u^2 + v^2).$$

Deduce that for $z, w \in \mathbb{C}$ we have

$$|z + w| \leq |z| + |w|.$$

EXERCISE 1.4. Find all complex solutions of the equations

$$z^2 = 1, \ z^4 = 1, \ z^8 = 1.$$

Write the solutions in the form $z = a + ib$ with $a, b \in \mathbb{R}$. Represent them on the complex plane.

EXERCISE 1.5. Solve in $\mathbb{C}$ the equation

$$e^z = 1.$$

You will find infinitely many solutions. Represent them on the complex plane.

EXERCISE 1.6. Solve the equation

$$z^3 + z^2 + z + 1 = 0$$

in $\mathbb{C}$.

EXERCISE 1.7. Let $z = 2e^{i\sqrt{3}}$. Find

$$\bar{z}, \ |z|, \ \mathrm{Re}z, \ \mathrm{Im}z.$$

EXERCISE 1.8. Consider the second order, linear, omogeneous ordinary differential equation

$$u'' + 4u' + 5u = 0. \tag{1.16}$$

(i) Find all the **real valued** solutions $\mathbb{R} \xrightarrow{u} \mathbb{R}$ of (1.16).
(ii) Find the (unique) solution of (1.16) satisfying $u(0) = 1$, $u'(0) = 2$.

EXERCISE 1.9. For $j \in \mathbb{Z}$ and $\mathbb{Z} \xrightarrow{f} \mathbb{C}$, let $\mathbb{Z} \xrightarrow{Sf} \mathbb{C}$ be defined as

$$(Sf)(j) = f(j-1). \tag{1.17}$$

The operation $f \mapsto Sf$ is the **unit forward shift**.

For $m \in \mathbb{Z}$, let $f_m(j) = e^{\frac{2\pi imj}{N}}$. Show that

$$Sf_m = e^{-\frac{2\pi im}{N}} f_m.$$

In other terms, f_m is an eigenvector for S, relative to the eigenvalue $e^{-\frac{2\pi im}{N}}$.

EXERCISE 1.10. Out of the solution of $z^{15} = 1$, how many are also solution of $z^5 = 1$? And of $z^3 = 1$? Draw them on the complex plane.

EXERCISE 1.11. Show that if $j = 1, 2, \dots, N-1$ and $z_j = e^{\frac{2\pi ij}{N}}$, then

$$1 + z_j + z_j^2 + \cdots + z_j^{N-1} = 0.$$

2. Inner product spaces

2.1. The inner product on $\mathbb{C}^N$. Let $\mathbb{C}^N$ be the set of the *column vectors*

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \dots \\ z_N \end{pmatrix} \tag{1.18}$$

with $z_1, \dots, z_N \in \mathbb{C}$. We can think (and programmers in fact think) of the "array" z as a function $\{0, \dots, N\} \xrightarrow{z} \mathbb{C}$, $j \mapsto z_j$. We might replace $\{0, \dots, N\}$ by $\mathbb{Z}_N = \{0, 1, \dots, N-1\}$, in which case $z = \begin{pmatrix} z_0 \\ z_1 \\ \dots \\ z_{N-1} \end{pmatrix}$. We will use indices $1, \dots, N$ in this section, but we will switch to the other index set when we talk about the discrete Fourier transform. This should not cause any trouble.

The space $\mathbb{C}^N$ becomes a *vector space* on $\mathbb{C}$ under the operations of *sum* and *multiplication times a (complex) scalar*,

$$\begin{pmatrix} z_1 \\ z_2 \\ \dots \\ z_N \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_N \end{pmatrix} = \begin{pmatrix} z_1 + w_1 \\ z_2 + w_2 \\ \dots \\ z_N + w_N \end{pmatrix}, \quad a \begin{pmatrix} z_1 \\ z_2 \\ \dots \\ z_N \end{pmatrix} = \begin{pmatrix} az_1 \\ az_2 \\ \dots \\ az_N \end{pmatrix}, \tag{1.19}$$

where $z, w \in \mathbb{C}^N$ are vectors and $a \in \mathbb{C}$ is a scalar.

The *(standard) inner product* of two vectors in $\mathbb{C}^N$, $\mathbb{C}^N \times \mathbb{C}^N \xrightarrow{\langle\cdot,\cdot\rangle} \mathbb{C}$ is defined as

$$\langle z, w\rangle = \sum_{j=1}^{N} \overline{z_j} w_j. \tag{1.20}$$

PROPOSITION 1.12. *The inner product in $\mathbb{C}^N$ satisfies the following properties:*

(i) *for $\xi, z, w \in \mathbb{C}^N$ and $a, b \in \mathbb{C}$, we have*
$$\langle \xi, az + bw\rangle = a\langle \xi, z\rangle + b\langle \xi, w\rangle;$$
(ii) *for $z, w \in \mathbb{C}^N$,*
$$\langle w, z\rangle = \overline{\langle z, w\rangle};$$
(iii) *for $z \in \mathbb{C}^N$,*
$$\langle z, z\rangle \geq 0 \text{ and } \langle z, z\rangle = 0 \text{ if and only if } z = 0.$$

A fundamental property of the inner product is the following inequality.

THEOREM 1.13. *[Cauchy-Schwarz inequality] For $z, w \in \mathbb{C}^N$ we have*
$$|\langle w, z\rangle| \leq \langle z, z\rangle^{1/2}\langle w, w\rangle^{1/2}. \tag{1.21}$$
Equality holds if and only if z and w are linearly dependent.

We define

$$\|z\| := \langle z, z\rangle^{1/2}$$

to be the *norm (associated to $\langle z, w\rangle$)* of $z \in \mathbb{C}^N$. The Cauchy-Schwarz inequality becomes

$$|\langle w, z\rangle| \leq \|z\| \cdot \|w\|.$$

PROPOSITION 1.14. *The norm has the following properties.*

(i) *For $z \in \mathbb{C}^N$, $\|z\| \geq 0$ and $\|z\| = 0$ if and only if $z = 0$.*
(ii) *For $z \in \mathbb{C}^N$ and $a \in \mathbb{C}$, $\|az\| = |a| \cdot \|z\|$.*
(iii) *For $z, w \in \mathbb{C}^N$, $\|z + w\| \leq \|z\| + \|w\|$ (**triangle inequality**).*

The inner product in $\mathbb{C}^N$ can be interpreted in terms of of row×column products. For z in $\mathbb{C}^N$, a *column*, let

$$z^* = (\overline{z_1}, \ldots, \overline{z}_N) = \overline{z}^t, \tag{1.22}$$

where w^t is the *transpose* of $w \in \mathbb{C}^N$, which is a *row*. Then,

$$\langle z, w\rangle = z^* w. \tag{1.23}$$

2.2. Matrices and linear operators in $\mathbb{C}^N$. The *standard basis* of $\mathbb{C}^N$ is $\{\delta_1, \ldots, \delta_N\}$, where

$$\delta_j = \begin{pmatrix} 0 \\ \ldots \\ 0 \\ 1 \\ 0 \\ \ldots \\ 0 \end{pmatrix}, \tag{1.24}$$

where 1 occurs in the j^{th} position. We write $\delta_j = \delta_{N,j}$ when we have to be explicit about the dimension.

A *linear operator* $\mathbb{C}^N \xrightarrow{L} \mathbb{C}^N$ between vector spaces U, V on $\mathbb{C}$ is a map satisfying

$$L(az + bw) = aL(z) + b(w) \tag{1.25}$$

whenever $z, w \in \mathbb{C}^N$ and $a, b \in \mathbb{C}$. The linear operators $\mathbb{C}^N \xrightarrow{L} \mathbb{C}_M$ can be written as products times a matrix. Let $A_{jk} = \langle \delta_{M,j}, L(\delta_{N,k})\rangle_{\mathbb{C}^M}$. Then, $L(z) = Az$, where

$$\begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1N} \\ A_{21} & A_{22} & \ldots & A_{2N} \\ & \ldots & & \\ A_{M1} & A_{M2} & \ldots & A_{MN} \end{pmatrix} \tag{1.26}$$

That is,

$$\begin{pmatrix} w_1 \\ w_2 \\ \ldots \\ w_M \end{pmatrix} = w = L(z) = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1N} \\ A_{21} & A_{22} & \ldots & A_{2N} \\ & \ldots & & \\ A_{M1} & A_{M2} & \ldots & A_{MN} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \ldots \\ z_N \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^N A_{1j} z_j \\ \sum_{j=1}^N A_{2j} z_j \\ \ldots \\ \sum_{j=1}^N A_{Mj} z_j \end{pmatrix}. \tag{1.27}$$

We denote by $\mathcal{M}_{M,N}(\mathbb{C})$ the set of the matrices with M rows and N colums. We can write $A \in \mathcal{M}_{M,N}(\mathbb{C})$ as a column of M row vectors in $[\mathbb{C}^N]^*$, or as a row of N column vectors in $\mathbb{C}^M$:

$$A = \begin{bmatrix} A^1 \\ A^2 \\ \ldots \\ A^M \end{bmatrix} = [A_1|A_2|\ldots|A_N], \tag{1.28}$$

with $A^1, \ldots, A^M \in [\mathbb{C}^N]^*$ and $A_1, \ldots, A_M \in \mathbb{C}^M$.

Besides the operations of of sum and multiplication times a scalar in $\mathcal{M}_{M,N}(\mathbb{C})$, we have the *matrix product*

$$\mathcal{M}_{M,N}(\mathbb{C}) \times \mathcal{M}_{N,P}(\mathbb{C}) \xrightarrow{\cdot} \mathcal{M}_{M,P}(\mathbb{C}),$$

which is computed following the row×column procedure. That is,

$$\begin{bmatrix} A^1 \\ A^2 \\ \ldots \\ A^M \end{bmatrix} [B_1|B_2|\ldots|B_P] = \begin{bmatrix} A^1B_1 & A^1B_2 & \ldots & A^1B_P \\ A^2B_1 & A^2B_2 & \ldots & A^2B_P \\ & \ldots & & \\ A^MB_1 & A^MB_2 & \ldots & A^MB_P \end{bmatrix}, \tag{1.29}$$

where $A^1, A^2, \ldots, A^M$ are rows in $[\mathbb{C}^N]^*$, and $B_1, B_2, \ldots, B_P$ are columns in $\mathbb{C}^N$.

Given a matrix $A \in \mathcal{M}_{M,N}(\mathbb{C})$ as in (1.28), the *rank* of A is the maximum number of linearly independent columns among $A_1, \dots, A_N$, which coincides with the maximum number of independent rows among $A^1, \dots, A^M$.

The case $M = N$ is particularly interesting,

$$\mathcal{M}_{N,N}(\mathbb{C}) \times \mathcal{M}_{N,N}(\mathbb{C}) \xrightarrow{\cdot} \mathcal{M}_{N,N}(\mathbb{C}).$$

A matrix $A \in \mathcal{M}_{N,N}(\mathbb{C})$ is *invertible* if there is $A^{-1} \in \mathcal{M}_{N,N}(\mathbb{C})$, its *inverse*, such that

$$A^{-1}A = I$$

is the *identity matrix*. If A^{-1} exists, then $AA^{-1} = I$. The following are equivalent for $A \in \mathcal{M}_{N,N}(\mathbb{C})$:

(1) A is invertible;
(2) A has rank N;
(3) $\det A \neq 0$.

The *adjoint* L^* of $\mathbb{C}^N \xrightarrow{L} \mathbb{C}_M$ is $\mathbb{C}^M \xrightarrow{L^*} \mathbb{C}_N$ which satisfies

$$\langle L^* z, w\rangle_{\mathbb{C}^N} = \langle z, Lw\rangle_{\mathbb{C}^M}. \tag{1.30}$$

If L is represented by the $M \times N$ matrix A in (1.26), L^* is represented by the *adjoint matrix* A^*,

$$A^*_{jk} = \overline{A_{kj}}, \ A^* = \begin{pmatrix} \overline{A_{11}} & \overline{A_{21}} & \dots & \overline{A_{M1}} \\ \overline{A_{12}} & \overline{A_{22}} & \dots & \overline{A_{M2}} \\ & \dots & & \\ \overline{A_{1N}} & \overline{A_{2N}} & \dots & \overline{A_{MN}}. \end{pmatrix} \tag{1.31}$$

Suppose $M = N$. The operator $\mathbb{C}^N \xrightarrow{L} \mathbb{C}^N$ is *self-adjoint* if $L = L^*$. The operator L is self-adjoint if and only if the matrix representing it is self-adjoint, $A = A^*$. That is,

$$A_{kj} = \overline{A_{jk}}. \tag{1.32}$$

In particular, the elements on the diagonal are real numbers, $A_{jj} = \overline{A_{jj}}$.

2.3. Inner product spaces in general. In real life, our vector spaces do not typically come endowed with coordinates, as we shall see below, and we need to consider matters in a coordinate-free environment. Let $\mathbf{V}$ be a vector space over $\mathbb{C}$. A *inner product* on $\mathbf{V}$ is a map $\mathbf{V} \times \mathbf{V} \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{C}$ satisfying (i-iii) in proposition 1.12:

(i) for $\xi, z, w \in \mathbf{V}$ and $a, b \in \mathbb{C}$, we have

$$\langle \xi, az + bw\rangle = a\langle \xi, z\rangle + b\langle \xi, w\rangle;$$

(ii) for $z, w \in \mathbf{V}$,

$$\langle w, z\rangle = \overline{\langle z, w\rangle};$$

(iii) for $z \in \mathbf{V}$,

$$\langle z, z\rangle \geq 0 \text{ and } \langle z, z\rangle = 0 \text{ if and only if } z = 0.$$

As in the case $\mathbf{V} = \mathbb{C}^N$, we define the *norm* of $z \in \mathbf{V}$ to be

$$\|z\| = \langle z, z\rangle^{1/2}.$$

We say that z and w are *orthogonal*, $z \perp w$, if $\langle z, w\rangle = 0$.

LEMMA 1.15. *If $z \perp w$, then*

$$\|z+w\|^2 = \|z\|^2 + \|w\|^2.$$

PROOF.

$$\begin{aligned} \|z+w\|^2 &= \langle z+w, z+w\rangle \\ &= \langle z,z\rangle + \langle z,w\rangle + \langle w,z\rangle + \langle w,w\rangle \\ &= \|z\|^2 + \|w\|^2. \end{aligned}$$

□

The Cauchy-Schwarz inequality, theorem 1.13, and the properties of the norm in proposition 1.14 continue to hold.

THEOREM 1.16. *[Cauchy-Schwarz inequality] Let $z, w \in \mathbf{V}$. Then,*

$$|\langle z,w\rangle| \le \|z\| \cdot \|w\|,$$

with equality if and only if z and w are linearly dependent.

PROOF. If $z = 0$, the inequality holds because both sides of the inequality vanish (and $0, w$ are linearly dependent). Suppose $z \neq 0$. We first verify that $z \perp w - \frac{\langle z,w\rangle z}{\|z\|^2}$:

$$\begin{aligned} \left\langle z, w - \frac{\langle z,w\rangle z}{\|z\|^2} \right\rangle &= \langle z,w\rangle - \langle z,w\rangle \frac{\langle z,z\rangle}{\|z\|^2} \\ &= 0. \end{aligned}$$

Hence, $\frac{\langle z,w\rangle z}{\|z\|^2} \perp w - \frac{\langle z,w\rangle z}{\|z\|^2}$ and, by Pythagoras theorem,

$$\begin{aligned} \|w\|^2 &= \left\| \frac{\langle z,w\rangle z}{\|z\|^2} \right\|^2 + \left\| w - \frac{\langle z,w\rangle z}{\|z\|^2} \right\|^2 \\ &\le \left\| \frac{\langle z,w\rangle z}{\|z\|^2} \right\|^2 \\ &= \frac{|\langle z,w\rangle|^2}{\|z\|^2}, \end{aligned}$$

which gives the desired inequality. Equality holds if and only if $w - \frac{\langle z,w\rangle z}{\|z\|^2} = 0$, which holds if and only if w is a scalar multiple of z. □

If $\mathbf{V}$ is finite dimensional, and it has dimension N, then it possesses a *orthonormal basis* $\{f_1, \ldots, f_N\}$:

$$\langle f_j, f_k\rangle = \delta_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases} \tag{1.33}$$

The case of infinite dimensional vector spaces is the most interesting one in this more abstract formulation, and it is crucial in applications. The definition of orthonormal basis in the infinite dimensional case is subtle. The same way, linear operators on infinite dimensional inner product spaces present some complications. We will considered the infinite dimensional case later on, after we have some concrete and useful examples.

If $\mathbf{V}$ is complex vector space with inner product $\langle\cdot,\cdot\rangle$, we can express it in terms of an expression only containing norms, which goes under the name of *polarization identity*:

$$\langle z,w\rangle=\frac{1}{4}[\langle z+w,z+w\rangle-\langle z-w,z-w\rangle-i(\langle z+iw,z+iw\rangle-\langle z-iw,z-iw\rangle)], \tag{1.34}$$

i.e.

$$\langle z,w\rangle=\|z+w\|^2+\|z-w\|^2+-i(\|z+iw\|^2-\|z-iw\|^2).$$

Let's check it. With $\xi=\langle z,w\rangle$, we have:

$$\begin{aligned}4\cdot R.H.S. &= 2\xi+2\bar{\xi}+2\xi-2\bar{\xi}\\ &= 4\langle z,w\rangle=4\cdot L.H.S.\end{aligned}$$

2.4. Unitary matrices. A matrix $U\in\mathcal{M}_{N,N}(\mathbb{C})$ is *unitary* if and only if U is invertible and $U^{-1}=U^*$. In this case,

$$U^*U=UU^*=I.$$

The class of the unitary matrices is denoted by $\mathcal{U}_N$.

The importance of unitary matrices is that they preserve the inner product.

THEOREM 1.17. *The following are equivalent for a matrix $U\in\mathcal{M}_{N,N}(\mathbb{C})$.*
(i) *$U\in\mathcal{U}_N$ is unitary;*
(ii) *$U=[U_1|\dots|U_N]$, where $U_1,\dots,U_N$ is a orthonormal basis of $\mathbb{C}^N$;*
(iii) *for all $z,w\in\mathbb{C}^N$, $\langle Uz,Uw\rangle=\langle z,w\rangle$;*
(iv) *for all $z\in\mathbb{C}^N$, $\|Uz\|=\|z\|$.*

PROOF. □

2.5. Inner products associated to integrals. Let I be an interval in $\mathbb{R}$, which might be bounded, unbounded, and even the real line itself. We denote by $C_c(I)$ the space of the continuous functions $I\xrightarrow{f}\mathbb{C}$ such that f has *compact support*: there is a closed, bounded interval $[p,q]\subseteq I$ such that $f(t)=0$ if $t\in I\setminus[p,q]$. For $f,g\in C_c(I)$, define the L^2*-inner product*

$$\langle f,g\rangle_{L^2}:=\int_I\overline{f(t)}g(t)dt. \tag{1.35}$$

THEOREM 1.18. (i) *$\langle\cdot,\cdot\rangle_{L^2}$ is an inner product on $C_c(I)$;*
(ii) *the Cauchy-Schwarz inequality holds,*

$$\left|\int_I\overline{f(t)}g(t)dt\right|\le\left(\int_I|f(t)|^2dt\right)^{1/2}\left(\int_I|g(t)|^2dt\right)^{1/2}. \tag{1.36}$$

PROOF. (i) is left as an exercise. (ii) holds for all inner product spaces, as we have seen in the previous subsection.. □

The L^2*-norm*, that associated with the L^2 inner product, is

$$\|f\|_{L^2}:=\left(\int_I|f(t)|^2dt\right)^{1/2}.$$

It satisfies the properties of the norm we have seen in proposition 1.14.

A slight variation of the above concernes *periodic functions*. Let $T > 0$ be fixed. A function $\mathbb{R} \xrightarrow{f} \mathbb{C}$ is T*-periodic* if

$$f(t+T) = f(t) \tag{1.37}$$

for all t in $\mathbb{R}$. We denote by $C_{T-per}(\mathbb{R})$ the class of the continuous, T-periodic functions. When the period T is fixed, we simply write $C_{per}(\mathbb{R})$ or C_{per}. In $C_{T-per}(\mathbb{R})$ we define the inner product

$$\langle f, g \rangle_{L^2_{per}} = \int_0^T \overline{f(t)} g(t) dt. \tag{1.38}$$

Observe that, by periodicity,

$$\int_0^T \overline{f(t)} g(t) dt = \int_a^{a+t} \overline{f(t)} g(t) dt$$

for all a in $\mathbb{R}$.

The inner product we have introduced satisfies the properties of an inner product, hence the Cauchy-Schwarz inequality.

We can replace integral with sums. Consider *doubly infinite sequences* $\mathbb{Z} \xrightarrow{\varphi} \mathbb{C}$. The space $\ell^2(\mathbb{Z})$ is populated by those sequences for which

$$\|\varphi\|_{\ell^2} := \left(\sum_{n=-\infty}^{\infty} |\varphi(n)|^2 \right)^{1/2} < \infty. \tag{1.39}$$

Lemma 1.19. *If $\varphi, \psi \in \ell^2(\mathbb{Z})$, then the series*

$$\sum_{n=-\infty}^{\infty} \overline{\varphi(n)} \psi(n) \tag{1.40}$$

converges absolutely.

Proof. Since convergence is absolute, we can assume $\varphi, \psi \geq 0$. Fix any $\epsilon > 0$. Suppose $N \geq N(\epsilon)$ is large enough to have

$$\sum_{|n|=N+1,\dots,N+M} \varphi(n)^2 \leq \epsilon, \quad \sum_{|n|=N+1,\dots,N+M} \psi(n)^2 \leq \epsilon \tag{1.41}$$

whenever $N \geq N(\epsilon)$. Such $N(\epsilon)$ exists by Cauchy criterion for the convergence of series, because both $\sum_{n=-\infty}^{\infty} |\varphi(n)|^2$ and $\sum_{n=-\infty}^{\infty} |\varphi(n)|^2$ converge.

We estimate the remainder of the series in (1.40) making use of the Cauchy-Schwarz inequality:

$$\begin{aligned} \sum_{|n|=N+1,\dots,N+M} \varphi(n)\psi(n) &\leq \left(\sum_{|n|=N+1,\dots,N+M} \varphi(n)^2 \right)^{1/2} \left(\sum_{|n|=N+1,\dots,N+M} \psi(n)^2 \right)^{1/2} \\ &\leq \epsilon. \end{aligned}$$

by (1.41). Again by Cauchy's criterion, (1.40) converges. □

The space $\ell^2(\mathbb{Z})$ so becomes an inner product space.

We might replace $\mathbb{Z}$ by $\mathbb{N}$, the nonnegative integers. The reason we used here $\mathbb{Z}$ is that it is the right object when studying Fourier series.

It is noteworthy that all the inner product spaces introduced above *are not finite dimensional*! This is a curse and a blessing. It's a curse, because it makes things more complicated; it's a blessing because we have infinitely many "degrees of freedom" in order to describe reality and design devices.

Let's see this in the case of the space $\ell^2(\mathbb{Z})$. Consider the sequences δ_n $(n \in \mathbb{Z})$,

$$\delta_n(m) = \begin{cases} 1 \text{ if } m = n, \\ 0 \text{ if } m \neq n. \end{cases}$$

The infinite family of vectors $\{\delta_n : n \in \mathbb{Z}\}$ is an orthonormal system,

$$\langle \delta_n, \delta_m \rangle_{\ell^2} = \delta_m(n),$$

hence, *a fortiori*, $\ell^2(\mathbb{Z})$ does not have finite dimension.

EXERCISE 1.20. Show that the functions $e_n(t) = e^{2\pi i n t}$, $n \in \mathbb{Z}$, form an orthonormal system for $C_{1-per}(\mathbb{R})$ with respect to the inner product (1.38).

EXERCISE 1.21. Consider $\mathbb{C}^N \ni \begin{pmatrix} z_0 \\ z_1 \\ \dots \\ z_{N-1} \end{pmatrix}$. Show that the functions

$$e_n(j) = \frac{1}{\sqrt{N}} e^{2\pi \frac{nj}{N}}.$$

$n \in \{0, 1, \dots, N-1\}$, form an orthonormal **basis** for $\mathbb{C}^N$ with respect to the standard inner product.

2.6. Exercises.

EXERCISE 1.22. Show that an orthonormal basis for $\mathbb{C}^2$ is provided by the vectors

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{1.42}$$

EXERCISE 1.23. Show that an orthonormal basis for $\mathbb{C}^4$ is provided by the vectors

$$\frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; \frac{1}{2} \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}, \; \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \; \frac{1}{2} \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}. \tag{1.43}$$

EXERCISE 1.24. For $n \in \mathbb{Z}$, let $e_n(t) = e^{2\pi i n t}$.

(i) Compute $\|e_n\|_{L^2}$.
(ii) Compute $\langle e_m, e_n \rangle_{L^2}$ when $m \neq n$.

EXERCISE 1.25. For $n \geq 1$ integer let $s_n(t) = \sin(2\pi n t)$, and for $n \geq 0$ integer let $c_n(t) = \cos(2\pi n t)$, $[0, 1] \xrightarrow{s_n, c_n} \mathbb{R}$.

(i) Compute $\|c_n\|_{L^2}$.
(ii) Compute $\langle c_m, c_n\rangle_{L^2}$ and $\langle s_m, s_n\rangle_{L^2}$ when $m \neq n$.
(iii) Compute $\langle c_m, s_n\rangle_{L^2}$ for all m, n.

Hint. Use exercise 1.24.

3. Integrals

Transforms are defined as integrals, as we shall see, and knowing what is meant by *integral* is relevant. We do not go through the theory, but just sketch the main definitions. We will provide simplified definitions, which are anyhow equivalent to the usual ones. While telling the story, I am taking some freedom in historical terms, but not too much.

The content of this section is not strictly necessary to read and understand the notes. If meet some integral we can compute, it can be computed using the usual tools from calculus. If we meet a function whose integral can be computed as usual, its integral is the limit of some approximations of it. As a matter of fact, we will only compute very classical integrals.

3.1. Integral of continuous functions. We denote by $C[a,b]$ the space of the continuous functions $[a,b] \xrightarrow{f} \mathbb{C}$. Its *integral in the sense of Cauchy* (1823) is

$$\int_a^b f(t)dt := \lim_{n\to\infty} \sum_{j=1}^{2^n} f\left(a + (j-1)\frac{b-a}{2^n}\right)\frac{b-a}{2^n}. \tag{1.44}$$

We have here subdivided $[a,b]$ in 2^n intervals $I_{n.j} = \left[a + (j-1)\frac{b-a}{2^n}, a + j\frac{b-a}{2^n}\right]$, each having length $\frac{b-a}{2^n}$, we computed f at the lower endpoint, multiplied times the length of $I_{n.j}$, then summed up everything, in the end taking the limit as $n \to \infty$. Cauchy proved that the limit exists in $\mathbb{C}$, and this provided the first mathematically rigorous definition of an object whose study went back at least as far as Newton and Leibnitz. Cauchy was motivated by the needs of the function series introduced by Joseph Fourier in 1807 in his study of the heat equation.

3.2. Riemann integrable functions. Motivated as well by Fourier series, the young Bernhard Riemann (1854) observed that choosing the left endpoint of $I_{n,j}$ was an unnecessary restriction. For $f \in C[a,b]$,

$$\int_a^b f(t)dt := \lim_{n\to\infty} \sum_{j=1}^{2^n} f\left(t_{n.j}\right)\frac{b-a}{2^n}, \tag{1.45}$$

the limit being independent of the the choice of points $a + (j-1)\frac{b-a}{2^n} \leq t_{n,j} \leq a + j\frac{b-a}{2^n}$. That was jus the beginning: Riemann proceeded to show that the existence of the limit in (1.45), of course independently of $t_{n,j}$ in the way just said, holds if and only if f has *vanishing average oscillation* on $[a,b]$, in a sense that I'll make precise.

For each n, we have an *approximation from below* (worst case scenario) to the would-be integral,

$$s(f,[a,b],n) := \sum_{j=1}^{2^n} \inf_{t\in I_{n,j}} f(t)\frac{b-a}{2^n},$$

and an *approximation from above* (again, worst case scenario),

$$S(f,[a,b],n) := \sum_{j=1}^{2^n} \sup_{s\in I_{n,j}} f(s)\frac{b-a}{2^n}$$

It is easy to see that $s(f,[a,b],n)$ increases with n, while $S(f,[a,b],n)$ decreases. They tend to the same limit if and only if their difference goes to zero.

For $[a,b] \xrightarrow{f} \mathbb{R}$, let

$$\mathrm{Osc}(f, I_{n,j}) = \sup_{s\in I_{n,j}} f(s) - \inf_{t\in I_{n,j}} f(t)$$

be the *oscillation* of f on $I_{n,j}$. The *average oscillation* of f relative to the decomposition $[a,b] = I_{n,1} \cup \cdots \cup I_{n,2^n}$ is, by definition.

$$\begin{aligned} \text{Av-Osc}(f,[a,b],n) \quad &:= \quad \frac{1}{2^n}\sum_{j=1}^{2^n}\mathrm{Osc}(f,I_{n,j}) = \frac{1}{2^n}\sum_{j=1}^{2^n}[\sup_{s\in I_{n,j}} f(s) - \inf_{t\in I_{n,j}} f(t)] \\ &= \quad S(f,[a,b],n,t_{n,1},\ldots,t_{n,2^n}) - s(f,[a,b],n,t_{n,1},\ldots,t_{n,2^n}). \end{aligned}$$

The function f has *vanishing average oscillation* on $[a,b]$ if

$$\lim_{n\to\infty} \text{Av-Osc}(f,[a,b],n) = 0, \tag{1.46}$$

i.e. if the approximations from below and above have the same limit. This happens, it can be proved, if and only if the limit in (1.45) exists and it's independent of the $t_{n,j}$'s.

Basically, the vanishing of the oscillation means that f does not oscillate too much too often. A priori, f can not be bounded, since unbounded functions on bounded intervals must have infinite oscillation on some $I_{n,j}$, for some j.

Among functions with this properties are those which are continuous on $[a,b]$ (nothing new), those which are bounded and have finitely many points of discontinuities, monotone ones, and all functions that can be obtained by these using sums, multiplication times constants, products, and various other manipulations. For all of these functions, we said, (1.45) provides a definition of integral which is well defined (and does not depend on the choice of the $t_{n,j}$'s) and that it captures our intuition of what an integral is. The functions having vanishing average oscillation are universally called *Riemann integrable*.

A function which *is not* Riemann integrable is the so-called *Dirichlet function* $[0,1] \xrightarrow{f} \mathbb{R}$:

$$f(t) = \begin{cases} 1 & \text{if } t \in [0,1]\cap\mathbb{Q}, \\ 0 & \text{if } t \in [0,1]\setminus\mathbb{Q}. \end{cases}$$

Since $\mathbb{Q}$ is *dense* in $\mathbb{R}$ [1], $\mathrm{Osc}(f, I_{n,j}) = 1$ for all n, j, hence, $\text{Av-Osc}(f,[a,b],n) = 1$, does not vanish as $n \to \infty$.

This was not a problem as far as the *Fourier analysis* of a function was concerned, but it became a headache when mathematicians started working on *Fourier synthesis*, where functions which are not Riemann integrable popped out.

[1] That is, if $x < y$ are real, there is a rational q such that $x < q < y$

3.3. Lebesgue integral. In 1902, Henri Lebesgue took a different point of view, which I illustrate for a function $f \geq 0$ on $[a, b]$. Again, we want to approximate f from below, but working on the ordinates instead of the abscissas. Fix $n \geq 1$ and, for $1 \leq j \leq (2^n)^2 = 2^{2n}$, consider

$$E_{n,j} = \{t \in [a, b] : (j-1)/2^n < f(x) \leq j/2^n\}. \tag{1.47}$$

An approximation from below of the the integral of $f$would be

$$\sigma(f, [a, b], n) := \sum_{j=1}^{2^{2n}} \frac{j-1}{2^n} \text{Length}(E_{n.j}), \tag{1.48}$$

where Length(E) is the "length" of the subset E of $[a, b]$. If a reasonable notion of length can be given, it is easy to see that $\sigma(f, [a, b], n)$ increases with n, and at this point we let

$$\int_a^b f(t)dt := \lim_{n\to\infty} \sigma(f, [a, b], n), \tag{1.49}$$

which in the end Lebesgue showed to coincide with Riemann's definition, if f is Riemann integrable.

The elephant in the room is defining a reasonable notion of length, and showing that it has some useful properties, A few years before Lebesge, Émile Borel had developed a theory of "sets having zero length"[2], which was a good starting point.

For an open interval (c, d), indeed Length$(c, d) = d - c$. Let $E \subset \mathbb{R}$ be bounded, and considered all countable families $\{(c_l, d_l)\}_{l=1}^{\infty}$ of open interval which *cover* E,

$$E \subseteq \cup_{l=1}^{\infty} (c_l, d_l),$$

and the intervals are two by two disjoint. Define

$$\text{Length}(E) := \inf\{\sum_{l=1}^{\infty} \text{Length}(c_l, d_l) : \ \{(c_l, d_l)\}_{l=1}^{\infty} \text{ is an open cover of } E\}. \tag{1.50}$$

Lebesgue showed that, at least for a wide class of "constructible sets",[3]the following properties hold:

(1) Length$(\emptyset) = 0$;
(2) if $\{E_m\}_{m=1}^{\infty}$ are disjoint, then

$$\text{Length}\left(\cup_{m=1}^{\infty} E_m\right) = \sum_{m=1}^{\infty} \text{Length}(E_m).$$

With this good notion of length at hands, Lebesgue managed to define the integral (1.49) for the "constructible functions", and to prove some crucial limit theorems which turned out to have many consequences beyond the application to the spectral synthesis.

[2]He had characterized Riemann integrable functions as those having a set of discontinuities with zero length.

[3]The class of sets for which the stated properties hold contains (i) the open intervals; (ii) all sets which can be obtained by previously constructed ones by means of (ii1) countable unions, and (ii2) set differences. This class is called the *Borel class*.

CHAPTER 2

Discrete and Fast Fourier Transform

1. The Discrete Fourier transform

In programming, the Fourier transform is usually computed using a finite, discrete analog of it, in which the base group encoding "time" is the finite group $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$ of the residues modulo N.

The *Discrete Fourier Transform* (DFT) of a function $f : \mathbb{Z}_N \to \mathbb{C}$ is

$$\widehat{f}(n) = \mathcal{F}_N f(n) = \sum_{j=0}^{N-1} e^{2\pi i n j/N} f(j). \tag{2.1}$$

Recal that the numbers $z_j := e^{2\pi i n j/N} = z_1^j$, $j = 0, \ldots, N-1$, are the complex roots of $z^N = 1$.

The DFT is interesting in itself, and it has a number of applications to geometry, combinatorics, probability, and so on.[1]

We might interpret $\mathcal{F}_N : \mathbb{C}^N \to \mathbb{C}^N$ a linear transformation, represented by the $N \times N$ matrix having elements $\mathcal{F}_N(n, j) = e^{2\pi i n j/N}$ with respect to the canonical basis of $\mathbb{C}^N$.

Lemma 2.1. (i) *The adjoint $\mathcal{F}_N^*$ of $\mathcal{F}_N$ with respect to the standard Hermitian inner product on $\mathbb{C}^N$ is $\mathcal{F}_N^*(m, n) = \overline{\mathcal{F}_N(n, m)}$.*
(ii) *As matrices, $\mathcal{F}_N^* \mathcal{F}_N = \mathcal{F}_N \mathcal{F}_N^* = N \cdot I_N$, where I_N is the identity operator on $\mathbb{C}^N$.*
(iii) $\langle \mathcal{F}_N f | \mathcal{F}_N g \rangle_{\ell^2(\mathbb{Z}_N)} = N \langle f | g \rangle_{\ell^2(\mathbb{Z}_N)}$.

Proof. (i) is just the definition, (iii) is a restatement of (ii), and (ii) is a calculation:

$$\begin{aligned} . \left[\mathcal{F}_N^* \mathcal{F}_N \right](m, n) &= \sum_{j=0}^{N-1} \mathcal{F}_N^*(m, j) \mathcal{F}_N(j, n) \\ &= \sum_{j=0}^{N-1} e^{-2\pi i j m/N} e^{2\pi i n j/N} = \sum_{j=0}^{N-1} e^{2\pi i j(n-m)/N} \end{aligned}$$

[1] In view of its ubiquitous applications, there is a cornucopia of resources on DFT both on library shelves and online. Here are just a couple of items of interest:

- The Fourier Transform and its Applications by Brad Osgood of the Electrical Engineering Department of Stanford University, which is a mathematically precise introduction to Fourier Transform for those who need it in applications;
- *Fourier Analysis on Finite Groups and Applications* by Audrey Terras of University of California, San Diego, with a number of applications of DFT within and outside mathematics.

$$= \begin{cases} N \text{ if } m = n \\ \frac{e^{N2\pi i(n-m)/N}-1}{e^{2\pi i(n-m)/N}-1} = 0 \text{ if } m \neq n. \end{cases}$$

□

Different normalizations of the involved inner products serve the purpose of approximating different "infinite" versions of the Fourier transform. Here, we consider

$$\mathcal{F}_N : \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N; 1/N), \tag{2.2}$$

where the latter is defined by the inner product:

$$\langle \varphi | \psi \rangle_{\ell^2(\mathbb{Z}_N;1/N)} := \sum_{j=0}^{N-1} \overline{\varphi(j)} \psi(j) \frac{1}{N}.$$

We can then translate Lemma 2.1, and some of its easy consequences, into a proposition with the main properties of the DFT.

PROPOSITION 2.2. (i) *We have* **Plancherel formula**

$$\frac{1}{N} \sum_{j=0}^{N-1} |\widehat{f}(j)|^2 = \sum_{l=0}^{N-1} |f(l)|^2.$$

(ii) **DFT inversion formula** *is*

$$f(l) = \mathcal{F}_N^{-1}(\widehat{f})(l) = \frac{1}{N} \sum_{j=0}^{N-1} \widehat{f}(j) e^{2\pi i l j/N}.$$

(iii) *For* $f, g \in \ell^2(\mathbb{Z}_N)$*, let* $f * g(m) = \sum_{j=0}^{N-1} f(m-j) g(j)$*. Then,*

$$\widehat{f * g}(m) = \widehat{f}(m) \widehat{g}(m).$$

In the other direction, if for $\varphi, \psi \in \ell^2(\mathbb{Z}_N; 1/N)$ *we define* $f * g(m) = \frac{1}{N} \sum_{j=0}^{N-1} \varphi(m-j) \psi(j)$*, then*

$$\mathcal{F}_N^{-1}(\varphi * \psi) = \mathcal{F}_N^{-1}(\varphi) \mathcal{F}_N^{-1}(\psi).$$

(iv) *Let* $\delta_0(m) = \begin{cases} 1 \text{ if } m = 0 \\ 0 \text{ if } m \neq 0. \end{cases}$ *. Then,* $\mathcal{F}_N(\delta_0) = 1$ *and* $\mathcal{F}_N(1) = N\delta_0$*.*

EXERCISE 2.3. Verify the assertions in Proposition 2.2.

2. The Fast Fourier transform

The *complexity* of computing $\mathcal{F}_N f$, the number of multiplications we have to perform, is $O(N^2)$. The *Fast Fourier Transform* (FFT) is an algorithm which drastically reduces the complexity of the task. The algorithm was first published in An Algorithm for the Machine Calculation of Complex Fourier Series by James W. Cooley and John W. Tukey, Math. Compute., vol. 19, pp. 297-301, April 1965. [2]

[2] Like many things which are both clever and a few steps from basic mathematics, FFT is found, unpublished, in one of Gauss' notebooks (1805), then independently reproved a number of times. As in the case of Shannon's sampling formula, this piece of maths became universally famous after being rediscovered by scientists working in big, cutting-edge laboratories: Shannon at Bell Labs, Cooley and Tukey at IBM's Watson labs (and Tukey was at the time sitting in President

We describe it in the case when $N = 2^M$. Depending on its use, there are variants for N with many factors, and even for N prime[3]. The basic, recursive step is a simple calculation reducing the DFT for $N = 2^M$ to $N = 2^{M-1}$.

$$\begin{aligned}
\mathcal{F}_{2^M} f(m) &= \sum_{j=1}^{2^M-1} f(j) e^{-2\pi imj/2^M} \\
&= \sum_{l=0}^{2^{M-1}-1} f(2l) e^{-2\pi iml/2^{M-1}} + e^{-2\pi im/2^M} \sum_{l=0}^{2^{M-1}-1} f(2l+1) e^{-2\pi iml/2^{M-1}} \\
&= \sum_{l=0}^{2^{M-1}-1} \left[f(2l) + e^{-2\pi im/2^M} f(2l+1) \right] e^{-2\pi iml/2^{M-1}}. \qquad (2.3)
\end{aligned}$$

Let $e_M : \mathbb{Z}/2^{M-1}\mathbb{Z} \to \mathbb{C}$, $e_M(n) = e^{-2\pi in/2^M}$, so that

$$e^{-2\pi im/2^M} = \begin{cases} e_M(m) \text{ if } 0 \le m \le 2^{M-1} - 1, \\ -e_M(m - 2^{M-1}) \text{ if } 2^{M-1} \le m \le 2^M - 1. \end{cases}$$

Set

$$\begin{aligned}
f_0(l) &= f(2l) \\
f_1(l) &= f(2l+1), \qquad (2.4)
\end{aligned}$$

with $f_0, f_1 : \mathbb{Z}/2^{M-1}\mathbb{Z}$. Equation (2.3) can be written in several useful ways. If $E_+ = \{m : \ 0 \le m \le 2^{M-1} - 1\}$ and $E_- = \{m : \ 2^{M-1} \le m \le 2^M - 1\}$,

$$\begin{aligned}
\mathcal{F}_{2^M} f &= \mathcal{F}_{2^{M-1}} f_0 + \chi_{E_+} e_M \mathcal{F}_{2^{M-1}} f_1 - \chi_{E_-} e_M \mathcal{F}_{2^{M-1}} f_1 \\
&= \begin{pmatrix} \mathcal{F}_{2^{M-1}} f_0 + \mathcal{F}_{2^{M-1}} f_1 e_M \\ \mathcal{F}_{2^{M-1}} f_0 - \mathcal{F}_{2^{M-1}} f_1 e_M \end{pmatrix} \\
&= \begin{pmatrix} I_{M-1} & e_M I_{M-1} \\ I_{M-1} & -e_M I_{M-1} \end{pmatrix} \begin{pmatrix} \mathcal{F}_{2^{M-1}} f_0 \\ \mathcal{F}_{2^{M-1}} f_1 \end{pmatrix}, \qquad (2.5)
\end{aligned}$$

where I_{M-1} is the identity matrix in $\mathbb{C}^{2^{M-1}}$.

Let's pause a moment to see what we have. Equation (2.5) tells us how to compute $\mathcal{F}_{2^M} f$, provided we already know $\mathcal{F}_{2^{M-1}} f_0$ and $\mathcal{F}_{2^{M-1}} f_1$. In (2.5), 2^M multiplications have to be performed, having as factors e_M, and $\mathcal{F}_{2^{M-1}} f_0$ or $\mathcal{F}_{2^{M-1}} f_1$.

Recursively, we reduce the calculation of $\mathcal{F}_{2^M}$ to 2 calculations of $\mathcal{F}_{2^{M-1}}$, then to 2^2 calculations of $\mathcal{F}_{2^{M-2}}$, etcetera. In the last step, we have 2^{M-1} calculations of

$$\mathcal{F}_2 \varphi = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \varphi(0) \\ \varphi(1) \end{pmatrix}.$$

The FFT algorithm is obtained by running the procedure backwards: compute first $\mathcal{F}_{2^1}$, feed it into $\mathcal{F}_{2^2}$ using (2.5) with $M = 2$, etcetera. Observe that the algorithm, only requires $N = 2^M$ numbers to be stored at any given time.

> Exercise 2.4. Rewrite the FFT algorithm as a product decomposition of the matrix representing $\mathcal{F}_{2^M}$ when $M = 2$, then for all M.

Kennedy's Science Advisory Committee). Feel free to draw your own conclusions on "all math people are on an equal footing". The fact is that what counts is not just the mathematical item *per se*, but also the context and the audience, which gives it potential for growth and applications.

[3] Discrete Fourier transforms when the number of data samples is prime by C.M. Rader Proceedings of the IEEE (Volume: 56, Issue: 6, June 1968)

For $N = 2^M$, let $C(N)$ be the *computational complexity* of the FFT: the number of multiplications which are needed to compute $\mathcal{F}_N(f)$ using the algorithm we have just described. We have:

$$C(2^M) = 2 \cdot C(2^{M-1}) + 2^M. \tag{2.6}$$

In fact, as observed above, in (2.5) we have to perform 2^M multiplications to reduce $\mathcal{F}_{2^M}$ to $\mathcal{F}_{2^{M-1}}$, and the transform $\mathcal{F}_{2^{M-1}}$ must be performed twice. We then iterate (2.6),

$$\begin{aligned} C(2^M) &= 2 \cdot C(2^{M-1}) + 2^M \\ &= 2 \cdot [2 \cdot C(2^{M-2}) + 2^{M-1}] + 2^M = 2^2 \cdot C(2^{M-2}) + 2 \cdot 2^M \\ &\dots \\ &= 2^{M-1} C(2) + (M-1)2^M \\ &= M 2^M = N(\log_2 N). \end{aligned}$$

Proposition 2.5. *The FFT has complexity $N \log_2 N$ (when $N = 2^M$).*

CHAPTER 3

Fourier series

1. The Fourier coefficients and their basic properties

A function $\mathbb{R} \xrightarrow{f} \mathbb{C}$ is *T-periodic* (with $T > 0$) if

$$f(t+T) = f(t)$$

whenever $t \in \mathbb{R}$. The number $T > 0$ is *a period* for f. Observe that if f is T-periodic, then it is $2T$-priodic, $3T$-periodic, etcetera. For instance, the following functions are 1-periodic:

$$f(t) = \cos(2\pi t),\ \sin(2\pi t),\ e^{2\pi i t}.$$

All functions obtained by T-periodic functions by means of the usual operations and compositions with other functions are T-periodic. For instance,

$$f(t) = \cos(2\pi t)e^{\sin(2\pi t)}$$

is 1-periodic.

A function $[0,1) \xrightarrow{\varphi} \mathbb{C}$ can be extended to a function $\mathbb{R} \xrightarrow{\varphi} \mathbb{C}$ which is 1-periodic by letting

$$\varphi(t+n) = \varphi(t) \tag{3.1}$$

whenever $t \in [0,1)$ and $n \in \mathbb{Z}$.

Let $\mathbb{R} \xrightarrow{\varphi} \mathbb{C}$ be a 1-periodic function such that

$$\|\varphi\|_{L^1} := \int_0^1 |\varphi(t)|dt < \infty. \tag{3.2}$$

For $n \in \mathbb{Z}$, let

$$\widehat{\varphi}(n) = \int_0^1 \varphi(t)e^{-2\pi i n t}dt \tag{3.3}$$

be the n^{th} *Fourier coefficient* of f. We can think of $\widehat{\varphi}$ as of a function defined on the integers,

$$\mathbb{Z} \xrightarrow{\widehat{\varphi}} \mathbb{C},\ n \mapsto \widehat{\varphi}(n). \tag{3.4}$$

For $a \in \mathbb{R}$ and $\mathbb{R} \xrightarrow{\varphi} \mathbb{C}$, let

$$(\tau_a\varphi)(t) := \varphi(t-a). \tag{3.5}$$

The operator $\tau_a : \varphi \mapsto \tau_a\varphi$ is the *shift* by a, and it has the effect of translating forward by a units the graph of the the function φ (it is a *delay* if $a > 0$). The function $\tau_a\varphi$ retains many properties of φ. In particular: φ is 1-periodic if and only if $\tau_a\varphi$ is; φ is continuous if and only if $\tau_a\varphi$ is.

It is clear that $\tau_a\varphi$ is T-periodic if and only $\tau_a\varphi$ is. If φ is 1-periodic, in particular, then

$$\int_0^1 |\varphi(t)|dt < \infty \text{ if and only if } \int_0^1 |(\tau_a\varphi)(t)|dt < \infty$$

(in fact, more is true: $\int_0^1 |\varphi(t)|dt = \int_0^1 |(\tau_a\varphi)(t)|dt$).

PROPOSITION 3.1. *[Basic properties of the Fourier coefficients] Let* $\mathbb{R} \xrightarrow{\varphi,\psi} \mathbb{C}$ *be* 1*-periodic. Then, the following properties hold.*

(i) *If* $a, b \in \mathbb{C}$, *then* $\widehat{a\varphi + b\psi}(n) = a\widehat{\varphi}(n) + b\widehat{\psi}(n)$.

(ii) *If* φ *is differentiable on* $\mathbb{R}$, *then*

$$\widehat{\varphi'}(n) = 2\pi in\widehat{\varphi}(n). \tag{3.6}$$

(iii) $\widehat{\tau_a\varphi}(n) = e^{2\pi ina}\widehat{\varphi}(n)$.

PROOF. □

The *convolution* $\varphi * \psi$ of two 1-periodic functions is defined as

$$(\varphi * \psi)(t) = \int_0^1 \varphi(t-s)\psi(s)ds. \tag{3.7}$$

THEOREM 3.2. *If* $\mathbb{R} \xrightarrow{\varphi,\psi} \mathbb{C}$ *are* 1*-periodic and* $n \in \mathbb{Z}$, *then*

$$\widehat{\varphi * \psi}(n) = \widehat{\varphi}(n)\widehat{\psi}(n). \tag{3.8}$$

PROOF. □

The crucial feature of convolution is that is commutes with time shifts.

THEOREM 3.3. *Suppose* $\mathbb{R} \xrightarrow{\varphi,\psi,\chi} \mathbb{C}$ *are* 1*-periodic and their modulus is integrable on* $[0, 1)$.

(i) *If* $a, b \in \mathbb{C}$, *then* $(a\varphi + b\psi) * \chi = a(\varphi * \chi) + b(\psi * \chi)$.

(ii) $\varphi * \psi = \psi * \varphi$.

(iii) $(\varphi * \psi) * \chi = \varphi * (\psi * \chi)$.

(iv) $\tau_a(\varphi * \psi) = (\tau_a\varphi) * \psi = \varphi * (\tau_a\chi)$.

(v) *If* φ *has continuous derivative, then* $(\varphi*)' = \varphi' * \psi$.

PROOF. □

2. Spectral analysis and synthesis

THEOREM 3.4. *Suppose* $[0, 1)$ *is square-summable,*

$$\|\varphi\|_{L^2} := \left(\int_0^1 |\varphi(t)|^2 dt\right)^{1/2} < \infty. \tag{3.9}$$

Then, the following hold.

(i) **Plancherel identity**

$$\int_0^1 |\varphi(t)|^2 dt = \sum_{n=-\infty}^{+\infty} |\widehat{\varphi}(n)|^2. \tag{3.10}$$

More generally, if also $\|\psi\|_{L^2} < \infty$*, then*

$$\int_0^1 \overline{\varphi(t)}\psi(t) dt = \sum_{n=-\infty}^{+\infty} \overline{\widehat{\varphi}(n)}\widehat{\psi}(n). \tag{3.11}$$

(ii) L^2**-convergence of the Fourier series**, *or* **Fourier analysis of a periodic function**. *We have that*

$$\varphi(t) =_{L^2} \sum_{n=-\infty}^{+\infty} \widehat{\varphi}(n) e^{2\pi i n t},$$

where the equality $=_{L^2}$ *in the* L^2 *sense means that*

$$0 = \lim_{N\to\infty} \int_0^1 \left| f(t) - \sum_{n=-N}^{+N} \widehat{\varphi}(n) e^{2\pi i n t} \right|^2 dt. \tag{3.12}$$

PROOF. □

In the other direction we have the following.

THEOREM 3.5. *Let* $\{a_n\}_{n=-\infty}^{+\infty}$ *be a two-sided infinite sequence of numbers in* $\mathbb{C}$ *such that*

$$\|\{a_n\}\|_{\ell^2} := \left(\sum_{n=-\infty}^{+\infty} |a_n|^2 \right)^{1/2} . \tag{3.13}$$

Then, there exists a unique $[0,1) \xrightarrow{\varphi} \mathbb{C}$ *such that*

$$\widehat{\varphi}(n) = a_n \text{ for all } n \in \mathbb{Z}. \tag{3.14}$$

Moreover, $\varphi \in L^2[0,1)$ *and*

$$\int_0^1 |\varphi(t)|^2 dt = \sum_{n=-\infty}^{+\infty} |a_n|^2,$$

and

$$\varphi(t) =_{L^2} \sum_{n=-\infty}^{+\infty} a_n e^{2\pi i n t},$$

ie

$$0 = \lim_{N\to\infty} \int_0^1 \left| f(t) - \sum_{n=-N}^{+N} a_n e^{2\pi i n t} \right|^2 dt.$$

3. Exercises

EXERCISE 3.6. For any of the functions $f = f(t)$ you will find below, $f : [-1/2, 1/2) \to \mathbb{R}$:

(i) compute the Fourier coefficients $\widehat{f}(n)$, $n \in \mathbb{Z}$;

(ii) write down the partial Fourier series

$$s_N(f)(x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{2\pi int}$$

and the complete Fourier series

$$s_\infty(f)(t) = \sum_{n=-\infty}^{+\infty} \widehat{f}(n) e^{2\pi int};$$

(iii) write explicitly in which sense $s_n(f)$ converges to f in $L^2[-1/2, 1/2)$;

(iv) do we have pointwise convergence,

$$\lim_{N\to\infty} s_N(t) = f(t)$$

for all $t \in [-1/2, 1/2)$?

(v) write explicitly $s_\infty(f)(t)$ for $0 \le t < 1$;

(vi) write $s_\infty(f)$ in terms of sines and cosines.

Here is the list of the functions.

(1) $f(t) = \begin{cases} \frac{1}{2a} \text{ if } |t| \le a, \\ 0 \text{ if } a < |t| \le 1/2, \end{cases}$ where $0 < a \le 1/2$ is a fixed parameter.

(2) $f(t) = |t|$.

(3) $f(t) = t$.

(4) $f(t) = \cos(\pi t)$.

(5) $f(t) = \sin(6\pi t)$.

(6) $f(t) = \begin{cases} 1 \text{ if } 0 \le x \le \frac{1}{4}, \\ 0 \text{ otherwise.} \end{cases}$

(7) $f(t) = \begin{cases} \frac{1}{4} - |t| \text{ if } |t| \le \frac{1}{4}, \\ 0 \text{ otherwise.} \end{cases}$

(8) $f(t) = \frac{1}{1 - \frac{\sin(2\pi t)}{2}}$. **Hint.** Here it might be more convenient using a geometric series, rather than computing integrals.

CHAPTER 4

Fourier transforms

1. The Fourier transform on $L^1(\mathbb{R})$

We say that $\mathbb{R} \xrightarrow{f} \mathbb{C}$ belongs to the L^1-*class*[1] if

$$\|f\|_{L^1} := \int_{-\infty}^{+\infty} |f(x)|dx < \infty, \tag{4.1}$$

i.e. if f is absolutely integrable. We write $f \in L^1(\mathbb{R})$

For $\omega \in \mathbb{R}$, we define

$$\widehat{f}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt. \tag{4.2}$$

We also write $\widehat{f} = \mathcal{F}(f)$, the *Fourier transform* $\mathbb{R} \xrightarrow{\widehat{f}} \mathbb{C}$ of f. The following theorem has a list of basic properties of the Fourier transform.

Theorem 4.1. *Suppose $\mathbb{R} \xrightarrow{f,g} \mathbb{C}$ belong to $L^1(\mathbb{R})$.*

(i) *If $a, b \in \mathbb{C}$, then $\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$.*

(ii) *If $f * g$ is the* **convolution** *of f and g,*

$$(f * g)(t) = \int_{-\infty}^{+\infty} f(t-s)g(s)ds, \tag{4.3}$$

then

$$\widehat{f * g}(\omega) = \widehat{f}(\omega)\widehat{g}(\omega). \tag{4.4}$$

(iii) *For $a \in \mathbb{R}$, let*

$$\tau_a f(t) = f(t-a), \tag{4.5}$$

the **shift of f by** *a. Then,*

$$\widehat{\tau_a f}(\omega) = e^{-ia\omega}\widehat{f}(\omega) \tag{4.6}$$

is a **modulation** *of f.*

(iv) *If $r > 0$ and*

$$(D_r f)(t) = \frac{1}{r} f\left(\frac{t}{r}\right) \tag{4.7}$$

*is the (L^1-***normalized***)* **dilation** *of f, then*

$$\widehat{D_r f}(\omega) = \widehat{f}(r\omega). \tag{4.8}$$

[1] The letter L derives from Henri Lebesgue, the founder of the modern theory of integration.

(v) *If f is also differentiable and $f' \in L^1$, then*

$$\widehat{f'}(\omega) = i\omega \widehat{f}(\omega). \tag{4.9}$$

(vi) *If also $xf(x) = M_x f(x)$ belongs to L^1, then $\widehat{f}$ is differentiable and*

$$\frac{d\widehat{f}}{d\omega}(\omega) = -i \int_{-\infty}^{+\infty} xf(x)e^{-i\omega x}dx = -i\widehat{M_x f}(\omega). \tag{4.10}$$

Here are some examples of Fourier transforms.

(i) Let $f(t) = e^{-|t|}$. Then,

$$\widehat{f}(\omega) = \frac{2}{1+\omega^2}. \tag{4.11}$$

Here are the calculations.

$$\begin{aligned} \widehat{f}(\omega) &= \int_{-\infty}^{0} e^t e^{-i\omega t}dt + \int_0^{+\infty} e^{-t}e^{-i\omega t}dt \\ &= \left[\frac{e^{t(1-i\omega)}}{1-i\omega}\right]_{-\infty}^0 + \left[\frac{e^{t(-1-i\omega)}}{-1-i\omega}\right]_0^{+\infty} \\ &= \frac{1}{1-i\omega} + \frac{1}{1+i\omega} \\ &= \frac{(1+i\omega)+(1-i\omega)}{(1-i\omega)(1+i\omega)} \\ &= \frac{2}{1+\omega^2}. \end{aligned}$$

(ii) Let $f(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$. Then,

$$\widehat{f}(\omega) = e^{-\frac{\omega^2}{2}}. \tag{4.12}$$

Here is a proof. The function $f(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$ satisfies the 1^{st} order, linear, homogeneous differential equation

$$f'(t) + tf(t) = 0. \tag{4.13}$$

By items (v) and (vi) in theorem 4.1, $\widehat{f}$ satisfies

$$i\omega\widehat{f}(\omega) + t\frac{d\widehat{f}}{d\omega}(\omega) = 0, \tag{4.14}$$

which, after dividing by i, turns out to be the same as (4.13). Hence,

$$\widehat{f}(\omega) = Ce^{-\frac{\omega^2}{2}}.$$

In order to find the constant, we compute

$$C = \widehat{f}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}}dt = 1,$$

poiché $\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}dx = \sqrt{\pi}$ is the integral of the Gaussian.

(iii) Let $f(t) = \begin{cases} \frac{1}{2} \text{ if } |t| \le 1 \\ 0 \text{ if } |t| > 1. \end{cases}$. Then,

$$\widehat{f}(\omega) = \mathrm{Sinc}(\omega) := \frac{\sin(\omega)}{\omega}. \tag{4.15}$$

Let's see the details,

$$\begin{aligned}\widehat{f}(\omega) &= \frac{1}{2}\int_{-1}^{+1} e^{-i\omega t}dt \\ &= \frac{1}{2}\left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-1}^{+1} = \frac{e^{i\omega}-e^{-i\omega}}{2i\omega} \\ &= \frac{\sin(\omega)}{\omega}.\end{aligned}$$

(iv) From (i-iii) above and item (iv) in theorem 4.1, for $a > 0$ we have the following transforms:

$$f(t) = \frac{1}{a}e^{-\frac{t}{a}} \implies \widehat{f}(\omega) = \frac{2}{1+a^2\omega^2}, \tag{4.16}$$

$$f(t) = \frac{1}{\sqrt{2\pi a}}e^{-\frac{t^2}{2a}} \implies \widehat{f}(\omega) = e^{-a\omega^2}, \tag{4.17}$$

$$f(t) = f(t) = \begin{cases} \frac{1}{2a} \text{ if } |t| \leq a \\ 0 \text{ if } |t| > a. \end{cases} \implies \widehat{f}(\omega) = \frac{\sin(a\omega)}{a\omega}. \tag{4.18}$$

(4.19)

THEOREM 4.2. *Let $f \in L^1(\mathbb{R})$. Then,*

(i) $\mathbb{R} \xrightarrow{\widehat{f}} \mathbb{C}$ *is continuous and bounded,*

$$|\widehat{f}(\omega)| \leq \|f\|_{L^1}; \tag{4.20}$$

(ii) *[Riemann-Lebesgue] $\widehat{f}$ vanishes at infinity,*

$$\lim_{\omega\to\pm\infty} \widehat{f}(\omega) = 0. \tag{4.21}$$

2. The Fourier transform on $L^2(\mathbb{R})$

What about the L^2 theory? The one from which we expect the best results, and which is dominant in applications? The problem here is that $L^2(\mathbb{R}) \not\subseteq L^1(\mathbb{R}) \not\subseteq L^1(\mathbb{R})$ (recall that $L^2[0,1] \subset L^1[0,1]$). Since the first inclusion fails, we have to find a way to give a meaning to the integral defining $\widehat{f}(\omega)$ when $f \in L^2(\mathbb{R})$. We let, for $f \in L^2(\mathbb{R})$ and $\omega \in \mathbb{R}$:

$$\widehat{f}(\omega) := \lim_{R\to\infty} \int_{-R}^{+R} f(t)e^{-i\omega t}dt, \tag{4.22}$$

provided that the limit exists.

It turns out that the limit, in fact, exists "for most" ω's. Setting $\widehat{f}(\omega) = 0$ if the limit does not exist, we have a function $\mathbb{R}$, which satisfiaes the usual formulas of the L^2 theory.

THEOREM 4.3. *Let $\mathbb{R} \xrightarrow{f,g} \mathbb{C}$ be two L^2 functions. Then the following properties hold.*

(i) **Plancherel.**

$$\int_{\mathbb{R}} |f(t)|^2 dt = \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{f}(\omega)|^2 d\omega. \tag{4.23}$$

(ii) **Plancherel for two functions.**

$$\int_{\mathbb{R}} \overline{f(t)} g(t) dt = \frac{1}{2\pi} \int_{\mathbb{R}} \overline{\widehat{f}(\omega)} \widehat{g}(\omega)\omega. \tag{4.24}$$

(iii) **Fourier inversion Formula**

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\omega) e^{it\omega} d\omega. \tag{4.25}$$

We have some other useful Fourier transforms. For instance, if $f(t) = \frac{1}{\pi} \frac{1}{1+t^2}$, then

$$\widehat{f}(\omega) = e^{-|\omega|}. \tag{4.26}$$

By the Fourier inversion formula, in fact,

$$\frac{1}{2} e^{-|\omega|} = \frac{1}{2} e^{-|-\omega|} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-it\omega}}{1+t^2} dt.$$

From (iv) in theorem 4.1, then, if $f(t) = \frac{1}{\pi} \frac{a}{a^2+t^2} = \frac{1}{\pi a} \frac{1}{+\left(\frac{t}{a}\right)^2}$,

$$\widehat{f}(\omega) = e^{-a|\omega|}. \tag{4.27}$$

Exercise 4.4. Compute the Fourier transforms of the following functions.

(i) $f(t) = e^{-t^2}$, $g(t) = te^{-\frac{t^2}{2}}$, $h(t) = t^2 e^{-\frac{t^2}{2}}$;
(ii) $f(t) = e^{-2|t|}$, $g(t) = te^{-|t|}$, $h(t) = t^2 e^{-|t|}$;
(iii) $f(t) = \frac{t}{1+t^2}$, $g(t) = \frac{t}{(1+t^2)^2}$, $h(t) = \frac{1}{4+t^2}$;
(iv) $f * g$, dove $f(t) = \begin{cases} 1 \text{ if } |t| \leq 1 \\ 0 \text{ if } |t| \geq 1 \end{cases}$ and $g(t) = e^{-|t|}$.
(v) $f(t) = \begin{cases} |t| - 1 \text{ if } |t| \leq 1 \\ 0 \text{ if } |t| \geq 1 \end{cases}$; $g(t) = \begin{cases} 1 + 2t \text{ if } -\frac{1}{2} \leq t \leq 0, \\ 1 - t \text{ if } 0 \leq t \leq 1, \\ 0 \text{ otherwise.} \end{cases}$

3. The Dirac delta

The *Dirac delta* δ_a at $a \in \mathbb{R}$ is a "generalized function" (in fact, it's not a function: it is a *measure*). It is defined by the property that, whenever φ is a **continuous function**, $\varphi \in C(\mathbb{R}, \mathbb{C})$,

$$\int_{-\infty}^{+\infty} \varphi(t)\delta_a(t) dt = \varphi(a). \tag{4.28}$$

No true function can satisfy this.

It is a usesful device when we carry out integrations by parts. Let's see the example the Dirac delta originated from. Let

$$H(t) = \begin{cases} 0 \text{ if } t < 0, \\ 1 \text{ if } t \geq 0 \end{cases} \tag{4.29}$$

be the *Heaviside function*[2]. The function H has a jump discontinuity at $t = 0$, $H(0^+) - H(0^-) = 1$, and it has vanishing derivative elsewhere.

Let $\varphi \in C_c(\mathbb{R}, \mathbb{C})$ be a continuous function with bounded support (i.e. there exists $R > 0$ such that $\varphi(t) = 0$ if $|t| \geq 0$). Then,

$$\begin{aligned}\int_{-\infty}^{+\infty} \varphi'(t)H(t)dt &= \int_0^{+\infty} \varphi'(t)dt = \varphi(+\infty) - \varphi(0) = -\varphi(0) \\ &= -\int_{-\infty}^{+\infty} \varphi'(t)\delta_0(t)dt.\end{aligned}$$

This is the formula we would get from integration by parts if we had $H'(t) = \delta_0(t)$: the variation of H is concentrated at 0. We can extend this.

PROPOSITION 4.5. *Let $F : \mathbb{R} \to \mathbb{C}$ be a function which is C^1 on $\mathbb{R} \setminus \{a_1, \ldots, a_n\}$, having left limits $F(a_i^-)$ and right limits $F(a_i^+)$ at each a_i, $a_1 < a_2 < \cdots < a_n$. Let $\Delta F(a_i) = F(a_i^+) - F(a_i^-)$ be the corresponding jump. Also suppose that F' is integrable on each interval (a_{i-1}, a_i).*

Then, for all $\mathbb{R} \xrightarrow{\varphi} \mathbb{C}$ continuous and such that $\lim_{t\to\pm\infty} \varphi(t) = 0$, we have the integration by parts formula:

$$-\int_{-\infty}^{+\infty} F(t)\varphi(t)dt = \int_{-\infty}^{+\infty} \{F'(t) + \sum_{j=1}^{n} \Delta F(a_j)\delta_{a_j}(t)\}\varphi(t)dt. \tag{4.30}$$

From the viewpoint of integration, that is F has *(distributional) derivative* F'_d,

$$F'_d(t) = F'(t) + \sum_{j=1}^{n} \Delta F(a_j)\delta_{a_j}(t). \tag{4.31}$$

PROOF. Suppose for simplicity that $\varphi(t) = 0$ for $|t| \geq R$, and we can choose $R > 0$ such that

$$a_0 := -R < a_1 < \cdots < a_n < R =: a_{n+1}.$$

We have then

$$\begin{aligned}-\int_{\mathbb{R}} F(t)\varphi'(t)dt &= -\sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} F(t)\varphi'(t)dt \\ &= -\sum_{i=1}^{n} \left([F(t)\varphi(t)]_{a_{i-1}}^{a_i} - \int_{a_{i-1}}^{a_i} F'(t)\varphi(t)dt \right) \\ &= -\sum_{i=1}^{n} \left(F(a_i^-) - F(a_{i-1}^+) - \int_{a_{i-1}}^{a_i} F'(t)\varphi(t)dt \right) \\ &= \sum_{i=1}^{n} \left(\Delta F(a_i) + \int_{a_{i-1}}^{a_i} F'(t)\varphi(t)dt \right) \\ &= \int_{-\infty}^{+\infty} \{F'(t) + \sum_{j=1}^{n} \Delta F(a_j)\delta_{a_j}(t)\}\varphi(t)dt.\end{aligned}$$

□

We have the following facts:

[2]Oliver Heaviside (1850-1825) was an electrical engineer and a mathematician.

(i) $\widehat{\delta_a}(\omega) = e^{-ia\omega}$;
(ii) $\delta_{a+b} = \delta_a * \delta_b$;
(iii) $\tau_a\delta_b = \delta_{a+b}$;
(iv) $D_r\delta_0 = \delta_0$, $D_r\delta_a = \delta_{ra}$.

4. The Uncertainty Principle

THEOREM 4.6. *[Uncertainty principle] Let $f \in L^2(\mathbb{R})$. Then,*

$$\frac{\pi}{2}\left(\int_{-\infty}^{+\infty} |f(t)|^2 dt\right)^2 \leq \int_{-\infty}^{+\infty} |tf(t)|^2 dt \int_{-\infty}^{+\infty} |\omega\widehat{f}(\omega)|^2 d\omega. \tag{4.32}$$

Equality holds if $f(t) = e^{-at^2}$ is a Gaussian ($a > 0$).

PROOF. We assume that $f, \widehat{f}$ are rel valued. The changes in the general case are not hard to make.

$$\begin{aligned}
\left(\int_{-\infty}^{+\infty} f(t)^2 dt\right)^2 &= \left(\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{t} 2f(s)f'(s)ds\right]dt\right)^2 \\
&= 4\left(-\int_{-\infty}^{+\infty} sf(s)f'(s)ds\right)^2 \\
&\leq 4\int_{-\infty}^{+\infty}[sf(s)]^2 ds \int_{-\infty}^{+\infty} f'(s)^2 ds \\
&= 4\int_{-\infty}^{+\infty}[sf(s)]^2 ds \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\widehat{f'}(\omega)|^2 d\omega \\
&= 4\int_{-\infty}^{+\infty}[sf(s)]^2 ds \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\omega\widehat{f}(\omega)|^2 d\omega \\
&= \frac{2}{\pi}\int_{-\infty}^{+\infty}[sf(s)]^2 ds \int_{-\infty}^{+\infty} |\omega\widehat{f}(\omega)|^2 d\omega.
\end{aligned}$$

□

5. Shannon's interpolation formula

THEOREM 4.7. *Suppose $\delta > 0$ and $g : \mathbb{R} \to \mathbb{C}$ is such that $\widehat{g}(\omega) = 0$ for $|\omega| > \frac{\pi}{\delta}$. Then, g can be reconstructed from the discrete sample $\{g(\delta n)\}_{n=-\infty}^{+\infty}$:*

$$g(\tau) = \sum_{n=-\infty}^{+\infty} g(\delta n)\frac{\sin\left(\pi\frac{\tau-n\delta}{\delta}\right)}{\pi\frac{\tau-n\delta}{\delta}}. \tag{4.33}$$

PROOF. Suppose $\widehat{f}(\omega) = 0$ for $|\omega| > 1/2$. Then,

$$\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{f}(\omega)e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-1/2}^{+1/2} \widehat{f}(\omega)e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-1/2}^{+1/2} \sum_{n=-\infty}^{+\infty} a_n e^{i\omega t}d\omega \\
& \quad \text{with } a_n = \int_{-1/2}^{+1/2} e^{-2\pi i n\omega}\widehat{f}(\omega)d\omega = 2\pi f(2\pi n) \\
&= \sum_{n=-\infty}^{+\infty} f(2\pi n)\int_{-1/2}^{+1/2} e^{i\omega t}e^{-2\pi i n\omega}d\omega \\
&= \sum_{n=-\infty}^{+\infty} f(2\pi n)\int_{-1/2}^{+1/2} e^{i(t-2\pi n)\omega}d\omega \\
&= \sum_{n=-\infty}^{+\infty} f(2\pi n)\left[\frac{e^{i(t-2\pi n)\omega} - e^{i(t-2\pi n)\omega}}{t-2\pi n}\right]_{t=-1/2}^{+1/2} \\
&= \sum_{n=-\infty}^{+\infty} f(2\pi n)\frac{\sin\left(\frac{t-2\pi n)}{2}\right)}{\frac{t-2\pi n}{2}}.
\end{aligned}$$

More generally, if $\widehat{g}(\omega) = 0$ for $|\omega| > a/2$, i.e. $\widehat{g}(a\omega) = 0$ for $|\omega| > 1/2$, then the calculation above holds for $f(t) = g(t/a)$, providing

$$g(t/a) = \sum_{n=-\infty}^{+\infty} g(2\pi n/a)\frac{\sin\left(\frac{t-2\pi n)}{2}\right)}{\frac{t-2\pi n}{2}},$$

i.e.

$$g(\tau) = \sum_{n=-\infty}^{+\infty} g(2\pi n/a)\frac{\sin\left(\frac{a\tau-2\pi n}{2}\right)}{\frac{a\tau-2\pi n}{2}}.$$

This is easier to read if $a = \frac{2\pi}{\delta}$:

$$g(\tau) = \sum_{n=-\infty}^{+\infty} g(\delta n)\frac{\sin\left(\pi\frac{\tau-n\delta}{\delta}\right)}{\pi\frac{\tau-n\delta}{\delta}}.$$

□

CHAPTER 5

Applications of Fourier theory to partial differential equations

1. The heat equation

1.1. The homogeneous heat equation on the real line. We consider the following problem. Given as *data* a bounded, continuous $\mathbb{R} \xrightarrow{\varphi} \mathbb{R}$ ($\mathbb{C}$-valued functions φ are covered by the method we will explain), we look for continuous $u = u(x,t)$, $\mathbb{R}\times \xrightarrow{u} \mathbb{R}$, such that it solves the *initial value problem*:

$$\begin{cases} \partial_t u(x,t) & = \frac{1}{2}\partial_{xx}u(x,t) \text{ for } (x,t) \in \mathbb{R}\times(0,\infty), \\ u(x,0) & = \varphi(x). \end{cases} \tag{5.1}$$

We move the problem on the frequency side by letting

$$\widehat{u}(\omega,t) = \int_{-\infty}^{+\infty} u(x,t)e^{-i\omega x}dx. \tag{5.2}$$

Using the formulas for the Fourier transform of a derivative, the problem (5.1) becomes:

$$\begin{cases} \partial_t \widehat{u}(\omega,t) & = -\frac{\omega^2}{2}\widehat{u}(\omega,t) \text{ for } (\omega,t) \in \mathbb{R}\times(0,\infty), \\ \widehat{u}(\omega,t) & = \widehat{\varphi}(\omega). \end{cases} \tag{5.3}$$

For ω fixed, let $y(t) = \widehat{u}(\omega,t)$, $[0\infty) \xrightarrow{y} \mathbb{C}$. The function y satisfies

$$\begin{cases} y'(t) & = -\frac{\omega^2}{2}y(t) \text{ for } t \in [0,\infty), \\ y(0) & = \widehat{\varphi}(\omega). \end{cases} \tag{5.4}$$

We know how to solve it:

$$y(t) = Ce^{-\omega^2 t}, \text{ with } C = y(0) = \widehat{\varphi}(\omega).$$

That is,

$$\widehat{u}(\omega,t) = \widehat{N_t}(\omega)\widehat{\varphi}(\omega), \tag{5.5}$$

where $\widehat{N_t}(\omega) = e^{-\omega^2 t}$, i.e.

$$\begin{aligned} N_t(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{\omega^2}{2}t}e^{i\omega x}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{\eta^2}{2}}e^{i\eta\sqrt{t}x}\frac{d\eta}{\sqrt{t}} \\ &= \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}, \end{aligned}$$

as we have seen when computing the Fourier transform of a Gaussian. We have proved the following.

THEOREM 5.1. *The solution of* (5.1) *is*

$$u(t,t) = N_t * \varphi(x) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-y)^2}{2t}}}{\sqrt{2\pi t}} \varphi(y)dy. \tag{5.6}$$

You might wonder what happens when we have $k\partial_{xx}$ with $k > 0$ instead of $\frac{1}{2}\partial_{xx}$.

THEOREM 5.2. *Let* $\mathbb{R} \xrightarrow{\varphi} \mathbb{R}$ *be continuous and bounded, and* $k > 0$. *Then, the solution of*

$$\begin{cases} \partial_t u(x,t) & = k\partial_{xx} u(x,t) \text{ for } (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) & = \varphi(x) \end{cases} \tag{5.7}$$

is

$$u(x,t) = M_t * \varphi(x), \text{ with } M_t(x) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}.$$

EXERCISE 5.3. Prove theorem 5.7.

1.2. The non-homogeneous heat equation on the real line.

THEOREM 5.4. *Let* $\mathbb{R} \times (0,+\infty) \xrightarrow{f} \mathbb{R}$ *be continuous and bounded, and* $k > 0$. *Then, the solution of*

$$\begin{cases} \partial_t v(x,t) & = \frac{1}{2}\partial_{xx} v(x,t) + f(x,t) \text{ for } (x,t) \in \mathbb{R} \times (0,\infty), \\ v(x,0) & = 0 \end{cases} \tag{5.8}$$

is

$$v(x,t) = \int_0^t \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-y)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} f(y,s)dyds.$$

PROOF. With $\widehat{v}(\omega,t)$ defined as above, the differential equation becomes

$$\partial_t \widehat{v}(\omega,t) = -\frac{\omega^2}{2}\widehat{v}(\omega,t) + \widehat{f}(\omega,t),$$

i.e., again writing $y(t) = \widehat{v}(\omega,t)$,

$$\begin{cases} y'(t) + \frac{\omega^2}{2} y(t) = \widehat{f}(\omega,t), \\ y(0) = 0. \end{cases} \tag{5.9}$$

After multiplying times $e^{\frac{\omega^2}{2}t}$, the equation becomes:

$$e^{\frac{\omega^2}{2}t}\widehat{f}(\omega,t) = e^{\frac{\omega^2}{2}t} y'(t) + e^{\frac{\omega^2}{2}t}\frac{\omega^2}{2} y(t) = \frac{d}{dt}\left(y(t)e^{\frac{\omega^2 t}{2}}\right).$$

Together with the condition $y(0) = 0$, we have then

$$y(t) = e^{-\frac{\omega^2}{2}t}\int_0^t e^{\frac{\omega^2}{2}s}\widehat{f}(\omega,s)ds = \int_0^t e^{-\frac{\omega^2}{2}(t-s)}\widehat{f}(\omega,s)ds. \tag{5.10}$$

Anti-transforming and taking into account that products are anti-transformed to convolutions,

$$\begin{aligned} v(x,t) &= \int_0^t N_{t-s}(x-y) f(y,s)ds \\ &= \int_0^t \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-y)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} f(y,s)dyds. \end{aligned}$$

□

We consider now the non-homogeneous equation with non-zero initial values:

$$\begin{cases} \partial_t w(x,t) &= \frac{1}{2}\partial_{xx} w(x,t) + f(x,t) \text{ for } (x,t) \in \mathbb{R} \times (0,\infty), \\ w(x,0) &= \varphi(x). \end{cases} \tag{5.11}$$

EXERCISE 5.5. Show that the solution of (5.11) is $w = u + v$, where u solves (5.1) and v solves (5.8).

Thus,

$$w(x,t) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-y)^2}{2t}}}{\sqrt{2\pi t}} \varphi(y)dy + \int_0^t \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-y)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} f(y,s)dyds.$$

2. The Laplace equation in a half-space

THEOREM 5.6. *Let* $\mathbb{R} \xrightarrow{\varphi} \mathbb{R}$ *be continuous and bounded, and consider the* **boundary value problem***:*

$$\begin{cases} 0 &= \partial_{xx} u(x,y) + \partial_{yy} u(x,y) \text{ for } (x,y) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= \varphi(x), \\ 0 &= \lim_{y\to+\infty} u(x,t). \end{cases} \tag{5.12}$$

Then, (5.12) has a unique soluzion which is continuous on $\mathbb{R} \times [0,+\infty$, *which is given by*

$$u(x,y) = P_y * \varphi(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y}{(x-u)^2 + y^2} \varphi(u)du. \tag{5.13}$$

The family of functions $\{P_y : y > 0\}$,

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2}, \tag{5.14}$$

is called the *Poisson kernel* for the half-plane.

PROOF. After a Fourier transforms in the x variable, letting $z(y) = \widehat{u}(\omega, y)$, (5.13) becomes

$$\begin{cases} z''(y) - \omega^2 z(y) = 0, \\ z(0) = \widehat{\varphi}(\omega), \\ \lim_{y\to\infty} z(y) = 0. \end{cases}$$

After solving

$$z(y) = A(\omega)e^{\omega y} + B(\omega)e^{-\omega y},$$

we see that the condition $\lim_{y\to\infty} z(y) = 0$ implies that

$$\begin{cases} A(\omega) = 0 \text{ and } B(\omega) = \widehat{\varphi}(\omega) \text{ if } \omega > 0, \\ A(\omega) = \widehat{\varphi}(\omega) \text{ and } B(\omega) = 0 \text{ if } \omega < 0. \end{cases}$$

Then

$$\widehat{u}(\omega, y) = z(y) = e^{-|\omega|y}\widehat{\varphi}(\omega).$$

The formula (5.6) follows, since, as we have seen, $\widehat{P_y}(\omega) = e^{-|\omega|y}$. □

3. The wave equation

3.1. The wave equation by means of Fourier transforms. We consider the following problem with data $\varphi, \psi : \mathbb{R} \to \mathbb{R}$:

$$\begin{cases} \partial_{tt}u(x,t) = c^2\partial_{xx}u(x,t), \\ u(x,0) = \varphi(x), \\ \partial_t u(x,t) = \psi(x), \end{cases} \tag{5.15}$$

and look for solutions $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$. After taking Fourier transforms we have

$$\begin{cases} \partial_{tt}\widehat{u}(\omega,t) + c^2\omega^2\widehat{u}(\omega,t) = 0, \\ \widehat{u}(\omega,0) = \widehat{\varphi}(\omega), \\ \partial_t\widehat{u}(\omega,t) = \widehat{\psi}(\omega). \end{cases} \tag{5.16}$$

The ordinary differential equation has solutions

$$\widehat{u}(\omega,t) = A(\omega)\cos(c\omega t) + B(\omega)\sin(c\omega t),$$

subject to

$$A(\omega) = \widehat{u}(\omega,0) = \widehat{\varphi}(\omega), \;\; c\omega B(\omega) = \partial_t\widehat{u}(\omega,t) = \widehat{\psi}(\omega),$$

i.e.

$$\widehat{u}(\omega,t) = \cos(c\omega t)\widehat{\varphi}(\omega) + \frac{\sin(c\omega t)}{c\omega}\widehat{\psi}(\omega). \tag{5.17}$$

We have to antitransform. Observe that

$$\cos(c\omega t) = \frac{\widehat{\delta_{ct}}(\omega) + \widehat{\delta_{-ct}}(\omega)}{2},$$
$$\frac{\sin(c\omega t)}{c\omega} = \frac{1}{2c}\widehat{\chi_{[-ct,ct]}}(\omega).$$

We obtain *D'Alambert's formula*

$$u(x,t) = \frac{\varphi(-ct) + \varphi(ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds. \tag{5.18}$$

3.2. A different approach to the wave equation. Another instructive way to look at the wave equation

$$\partial_{tt}u(x,t) = c^2\partial_{xx}u(x,t) \tag{5.19}$$

is by means of the change of variables:

$$\begin{cases} p = x - ct, \\ q = x + ct \end{cases}, \text{ i.e. } \begin{cases} x = \frac{q+p}{2}, \\ t = \frac{q-p}{2c} \end{cases}. \tag{5.20}$$

Let

$$v(p,q) := u(x,t) = u\left(\frac{q+p}{2}, \frac{q-p}{2c}\right), \tag{5.21}$$

Computing derivatives, we find that

$$\frac{\partial^2 v}{\partial p \partial q}(p,q) = \frac{1}{4c^2}\left(c^2 \frac{\partial^2 u}{\partial x^2}(x,t) - \frac{\partial^2 u}{\partial t^2}(x,t)\right), \tag{5.22}$$

where (x,t) depends on (p,q) as specified by (5.20). Thus, u is a solution of the wave equation if and only if

$$\partial_{pq} v(p,q) = 0. \tag{5.23}$$

This holds if and only if $\partial_q v(p,q) = \tilde{D}(q)$ for some function $\tilde{D}$, and the latter hold if and only if

$$v(p,q) = D(p) + E(q) \tag{5.24}$$

for some twice-differentiable functions $E, D : \mathbb{R} \to \mathbb{R}$. After replacing p, q according to (5.20), we have the following.

THEOREM 5.7. *The solutions of the wave equation* (5.19) *have the form*

$$u(x,t) = D(x-ct) + E(x+ct) \tag{5.25}$$

for some functions D, E.

The equation (5.25) has a graphic intepretation: $u(x,t)$ is the *superposition* (sum) of a wave $D(x-ct)$ traveling with speed c along the real axis, and a wave $E(x+ct)$ traveling in the opposite direction, with speed $-c$.

4. Exercises

EXERCISE 5.8. Let $c > 0$ be fixed. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous and bounded. Show that the only continuous solution $u(x,t) = u : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ to the problem:

$$\begin{cases} \partial_t u(x,t) = c\partial_x u(x,t) \text{ when } x \in \mathbb{R},\ t > 0, \\ u(x,0) = \varphi(t) \end{cases}$$

is

$$u(x,t) = \varphi(x+ct),$$

a wave traveling backward in time with speed c.

What is the solution if the equation is replaced by $\partial_t u(x,t) = -c\partial_x u(x,t)$?

EXERCISE 5.9. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous and bounded. Find a continuous solution $u(x,t) = u : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ to the problem:

$$\begin{cases} \partial_t u(x,t) = \partial_{xx} u(x,t) - u(x,t) \text{ when } x \in \mathbb{R},\ t > 0, \\ u(x,0) = \varphi(t). \end{cases}$$

EXERCISE 5.10. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous with $\lim_{x\to\pm\infty} \varphi(x) = 0$. Find a continuous solution $u(x,t) = u : \mathbb{R} \times [0,+\infty) \to \mathbb{R}$ to the problem:

$$\begin{cases} \partial_{xx} u(x,y) + 4\partial_{yy} u(x,y) = 0 \text{ when } x \in \mathbb{R},\ y > 0, \\ u(x,0) = \varphi(y), \\ \lim_{y\to+\infty} u(x,y) = 0. \end{cases}$$

CHAPTER 6

Mock exams

First ME

(i) Compute the Fourier coefficients of $\varphi * \psi$ where $[-1/2, +1/2] \xrightarrow{\varphi,\psi} \mathbb{C}$,

$$\varphi(t) = \begin{cases} 1 \text{ if } 0 \le t \le 1/4, \\ 0 \text{ otherwise} \end{cases},$$
$$\psi(t) = |t|.$$

Write down the Fourier series of $\varphi * \psi$.

(ii) Compute the Fourier transform of $f(t) = t^2 e^{-t^2}$. Use the result to compute:

(a) $\int_{-\infty}^{+\infty} t^2 e^{-t^2} dt$;

(b) the covariance

$$\int_{-\infty}^{+\infty} t^2 N_1(t) dt$$

of the Gaussian $N_1(t) = \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}$.

(iii) Find the solution $\mathbb{R} \times [0, \infty) \xrightarrow{u} \mathbb{R}$ of the following boundary problem:

$$\begin{cases} 0 = \partial_{xx} u(x, y) + \partial_{yy} u(x, y) \text{ for } (x, y) \in \mathbb{R} \times (0, \infty), \\ u(x, 0) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}, \\ 0 = \lim_{y \to +\infty} u(x, t). \end{cases} \tag{6.1}$$